

THE BARTLE–DUNFORD–SCHWARTZ AND THE DINCULEANU–SINGER THEOREMS REVISITED

FERNANDO MUÑOZ, EVE OJA, AND CÁNDIDO PIÑEIRO

ABSTRACT. Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Denote by $\mathcal{C}_p(\Omega, X)$ the space of p -continuous X -valued functions, $1 \leq p \leq \infty$. For operators $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ and $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$, we establish integral representation theorems with respect to a vector measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$, where Σ denotes the σ -algebra of Borel subsets of Ω . The first theorem extends the classical Bartle–Dunford–Schwartz representation theorem. It is used to prove the second theorem, which extends the classical Dinculeanu–Singer representation theorem, also providing to it an alternative simpler proof. For the latter (and the main) result, we build the needed integration theory, relying on a new concept of the q -semivariation, $1 \leq q \leq \infty$, of a vector measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$.

1. INTRODUCTION

Let X be a Banach space and let Ω be a compact Hausdorff space. The space of continuous functions from Ω into X (\mathbb{K} , respectively) is denoted by $\mathcal{C}(\Omega, X)$ ($\mathcal{C}(\Omega)$, respectively). We denote by Σ the σ -algebra of Borel subsets of Ω . The space of Σ -simple functions with values in X and the Banach space of bounded Σ -measurable functions with values in X (i.e., the space of functions from Ω into X which are the uniform limit of a sequence of Σ -simple functions) are denoted by $\mathcal{S}(\Sigma, X)$ and $\mathcal{B}(\Sigma, X)$, respectively. In the case $X = \mathbb{K}$, we abbreviate them to $\mathcal{S}(\Sigma)$ and $\mathcal{B}(\Sigma)$, respectively. It is well known that $\mathcal{C}(\Omega) \subset \mathcal{B}(\Sigma) \subset \mathcal{C}(\Omega)^{**}$ and, more generally, $\mathcal{C}(\Omega, X) \subset \mathcal{B}(\Sigma, X) \subset \mathcal{C}(\Omega, X)^{**}$ as closed subspaces.

Let Y be a Banach space and denote by $\mathcal{L}(X, Y)$ the Banach space of bounded linear operators from X into Y . Let $m : \Sigma \rightarrow Y$ be a vector measure of bounded semivariation. It is well known (see, e.g., [8, pp. 6, 56, 153]) that the (elementary Bartle) integral $\int_{\Omega}(\cdot) dm$ is defined on $\mathcal{B}(\Sigma)$. (The definition passes from characteristic functions to functions in $\mathcal{S}(\Sigma)$ by linearity and to functions in $\mathcal{B}(\Sigma)$ by density.) By the Bartle–Dunford–Schwartz representation theorem, for every operator $S \in \mathcal{L}(\mathcal{C}(\Omega), Y)$ there

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exists a unique vector measure $m : \Sigma \rightarrow Y^{**}$ of bounded semivariation such that $S\varphi = \int_{\Omega} \varphi dm$ for all $\varphi \in \mathcal{C}(\Omega)$. The vector measure m is called the *representing measure* of S .

In [20, Section 4], this representation was extended from $Y \cong \mathcal{L}(\mathbb{K}, Y)$ to $\mathcal{L}(X, Y)$. Namely, in the case when $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, a vector measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ of bounded semivariation was built so that $S\varphi = \int_{\Omega} \varphi dm$ for all $\varphi \in \mathcal{C}(\Omega)$ (the construction of m is recalled in Remark 2.12 below). We define a *representing measure* of $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ as a vector measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ of bounded semivariation which satisfies

$$S\varphi = \int_{\Omega} \varphi dm \text{ for all } \varphi \in \mathcal{C}(\Omega).$$

In Section 2, we extend the Bartle–Dunford–Schwartz theorem, in all its aspects, to this general setting (see Theorem 2.4). We also find a formula connecting the measure m and the *classical* representing measure $\mu : \Sigma \rightarrow \mathcal{L}(X, Y)^{**}$ of S (as given by the Bartle–Dunford–Schwartz theorem) (see Corollary 2.9).

Results of Section 2 are applied in Section 4 to revisit the classical Dinculeanu–Singer representation theorem. By this theorem, for every operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$, there exists a unique vector measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ such that

$$Uf = \int_{\Omega} f dm \text{ for all } f \in \mathcal{C}(\Omega, X),$$

where the existence of the above integral requires from the measure m that its 1-semivariation (in our terminology; see Section 3 and, in particular, Example 3.1 showing that the 1-semivariation coincides with the Gowurin–Dinculeanu semivariation) is bounded.

In Section 4, see Theorem 4.8, which is the main result of this paper, we extend the Dinculeanu–Singer theorem, in all its aspects, from $\mathcal{C}(\Omega, X)$ to the Banach space $\mathcal{C}_p(\Omega, X)$ of p -continuous X -valued functions, where $1 \leq p \leq \infty$ (studied in [19] and [20]; see Section 3 for the definition and needed properties), the spaces $\mathcal{C}_p(\Omega, X)$ being contained in $\mathcal{C}(\Omega, X) = \mathcal{C}_{\infty}(\Omega, X)$. However, this is not a routine extension: we do not follow the traditional proofs of the Dinculeanu–Singer theorem (see Remark 4.11), but we provide a handy alternative to them.

The scheme of our proof is very simple: for $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$, we consider the *associated operator* $U^{\#} \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, defined by

$$(U^{\#}\varphi)x = U(\varphi x), \quad \varphi \in \mathcal{C}(\Omega), \quad x \in X.$$

By the above, we already have the representing measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ of $U^\#$. And we show (see Theorem 4.3) that $Uf = \int_\Omega f dm$ for all $f \in \mathcal{C}_p(\Omega, X)$, meaning that our m is also a representing measure of U .

On the other hand, one easily shows (see Proposition 4.4) that a representing measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ of $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ is also a representing measure of $U^\#$. Therefore, since the representing measure of $U^\#$ is unique, also the representing measure of U is unique. Hence, in the classical case when $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$, we regain the classical representing measure from the Dinculeanu–Singer theorem. Moreover, for the first time in the literature, a general formula, connecting the representing measure m of U and the classical representing measure $\mu : \Sigma \rightarrow \mathcal{L}(X, Y)^{**}$ of $U^\#$, is given (see Corollary 4.6 and Remark 4.7).

In Section 3, since the integration on $\mathcal{C}_p(\Omega, X)$ requires from the measure more than just the boundedness of its semivariation (but less than the integration on $\mathcal{C}(\Omega, X)$), we build the needed theory. For this end, we introduce the concept of the *q-semivariation* of a vector measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ of bounded semivariation. This enables us to define an integral on $\mathcal{C}_p(\Omega, X)$ with values in Y^{**} , provided that the p' -semivariation of m is bounded.

Finally, Section 5 is devoted to prove some qualitative complements to Theorem 4.8, our extension of the Dinculeanu–Singer theorem, it uses results from the paper [20] by the authors and can be read just after Proposition 4.4.

Our notation is standard. Let $1 \leq p \leq \infty$, and denote by p' the conjugate index of p (i.e., $1/p + 1/p' = 1$ with the convention $1/\infty = 0$). We consider Banach spaces over the same, either real or complex, field \mathbb{K} . A Banach space X will be regarded as a subspace of its bidual X^{**} under the canonical isometric embedding $j_X : X \rightarrow X^{**}$. The closed unit ball of X is denoted by B_X . The Banach space of all *absolutely p-summable sequences* in X is denoted by $\ell_p(X)$ and its norm by $\|\cdot\|_p$. The Banach operator ideal of absolutely p -summing operators is denoted by $\mathcal{P}_p = (\mathcal{P}_p, \|\cdot\|_{\mathcal{P}_p})$, and $\mathcal{L} = (\mathcal{L}, \|\cdot\|)$ is, as usual, the Banach operator ideal of bounded linear operators. We denote the characteristic function of $E \in \Sigma$ by χ_E .

Our main reference to the vector measure theory is the book [8] by Diestel and Uhl. In particular, a *vector measure* $m : \Sigma \rightarrow X$ is a finitely additive X -valued set function. The *semivariation* of m on Ω is denoted by $\|m\|(\Omega)$ and defined as

$$\|m\|(\Omega) = \sup \left\| \sum_{E_i \in \Pi} \varepsilon_i m(E_i) \right\|,$$

where the supremum is taken over all finite partitions $\Pi = (E_i)_{i=1}^n$ of Ω and all finite systems $(\varepsilon_i)_{i=1}^n$ with $|\varepsilon_i| \leq 1$, $1 \leq i \leq n$, $n \in \mathbb{N}$ (see, e.g., [8, p. 4, Proposition 11]). If $\|m\|(\Omega) < \infty$, then m is called a *measure of bounded semivariation*. A vector measure $m : \Sigma \rightarrow X$ is *bounded* if its range is bounded in X . This happens if and only if m is of bounded semivariation (see, e.g., [8, p. 4, Proposition 11]). Therefore, a vector measure of bounded semivariation is often called a *bounded vector measure* (see, e.g., [8, p. 5]), and we shall mainly use this term below.

2. REPRESENTING MEASURE OF $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$

Let X and Y be Banach spaces and let Ω be a compact Hausdorff space.

Definition 2.1. Let $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. A *representing measure* of S is a bounded vector measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ which satisfies

$$S\varphi = \int_{\Omega} \varphi dm \quad \text{for all } \varphi \in \mathcal{C}(\Omega).$$

As was mentioned in the Introduction, a representing measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ exists for every operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. Since we are going to use such a measure, it would be good (but not crucial) to know that it is unique. We start by a general observation that will also be used in Section 3.

Let $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ be a bounded vector measure. Then, for every $x \in X$,

$$m_x := m(\cdot)x : \Sigma \rightarrow Y^{**}$$

is clearly a bounded vector measure. If $y^* \in Y^*$, then $x \otimes y^* \in \mathcal{L}(X, Y^{**})^*$ and for all $\varphi \in \mathcal{B}(\Sigma)$,

$$(1) \quad \left\langle \int_{\Omega} \varphi dm, x \otimes y^* \right\rangle = \int_{\Omega} \varphi d\mu_{x, y^*} = \langle y^*, \int_{\Omega} \varphi dm_x \rangle,$$

where $\mu_{x, y^*} := (x \otimes y^*)m$, because

$$\mu_{x, y^*}(E) = ((x \otimes y^*)m)(E) = \langle m(E), x \otimes y^* \rangle = \langle y^*, m(E)x \rangle = \langle y^*, m_x(E) \rangle, \quad E \in \Sigma.$$

Let $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. For every $x \in X$, define $S_x \in \mathcal{L}(\mathcal{C}(\Omega), Y)$ by

$$S_x \varphi = (S\varphi)x, \quad \varphi \in \mathcal{C}(\Omega).$$

Lemma 2.2. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ be a representing measure of an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. Then $m_x : \Sigma \rightarrow Y^{**}$ is the (classical) representing measure of the operator $S_x \in \mathcal{L}(\mathcal{C}(\Omega), Y)$ (given by the Bartle–Dunford–Schwartz theorem) and $\mu_{x, y^*} = S_x^* y^*$ for all $x \in X$ and $y^* \in Y^*$.*

Proof. For all $\varphi \in \mathcal{C}(\Omega)$, $x \in X$ and $y^* \in Y^*$, by (1),

$$\langle S_x \varphi, y^* \rangle = \langle (S\varphi)x, y^* \rangle = \langle S\varphi, x \otimes y^* \rangle = \langle y^*, \int_{\Omega} \varphi dm_x \rangle.$$

This shows that m_x is the representing measure of the operator $S_x \in \mathcal{L}(\mathcal{C}(\Omega), Y)$. Hence $S_x^* \in \mathcal{L}(Y^*, \mathcal{C}(\Omega)^*)$ and, by the Bartle–Dunford–Schwartz theorem, it is well known that $S_x^* y^* = \langle y^*, m_x(\cdot) \rangle = \mu_{x,y^*}$. \square

Proposition 2.3. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Then the representing measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ of an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ is unique.*

Proof. Let $m_1, m_2 : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ be two representing measures of an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, and let $\mu_{x,y^*}^i = (x \otimes y^*)m_i$, $i = 1, 2$. We know from Lemma 2.2 that $\mu_{x,y^*}^1 = S_x^* y^* = \mu_{x,y^*}^2$, giving that $\langle y^*, m_1(E)x \rangle = \langle y^*, m_2(E)x \rangle$, for all $E \in \Sigma$, $x \in X$, and $y^* \in Y^*$. This means that $m_1 = m_2$. \square

Recalling that $\mathcal{L}(\mathbb{K}, Y) \cong Y$, the following result extends the classical Bartle–Dunford–Schwartz theorem (see, e.g., [8, p. 152]) in all its aspects.

Theorem 2.4. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space.*

(a) *Every operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ has a unique representing measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$.*

(b) *Assume that $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then, there exists an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ such that m is its representing measure if and only if for all $x \in X$,*

$$\mu_{x,y^*} = \langle y^*, m_x(\cdot) \rangle \in \mathcal{C}(\Omega)^*, \quad y^* \in Y^*,$$

and the map $Y^ \rightarrow \mathcal{C}(\Omega)^*$, $y^* \mapsto \langle y^*, m_x(\cdot) \rangle$, is linear, bounded, and weak*-to-weak* continuous.*

*In this case, $m_x : \Sigma \rightarrow Y^{**}$ is the representing measure of the operator $S_x \in \mathcal{L}(\mathcal{C}(\Omega), Y)$ and $\mu_{x,y^*} = S_x^* y^*$ for all $x \in X$ and $y^* \in Y^*$, the equality $\|S\| = \|m\|(\Omega)$ holds, and the measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**}) = (X \hat{\otimes}_{\pi} Y^*)^*$ is weak*-countably additive.*

Proof. (a) The existence of a representing measure was proved in [20, Section 4]. The (simple) construction of this measure is recalled in Remark 2.12. Its uniqueness comes from Proposition 2.3.

(b) By Lemma 2.2, we know that $m_x : \Sigma \rightarrow Y^{**}$ is the representing measure of $S_x \in \mathcal{L}(\mathcal{C}(\Omega), Y)$. Then $S_x^* \in \mathcal{L}(Y^*, \mathcal{C}(\Omega)^*)$ is weak*-to-weak*

continuous. Also, by Lemma 2.2, $S_x^* y^* = \mu_{x,y^*} = \langle y^*, m_x(\cdot) \rangle$. This proves the “only if” part.

For the “if” part, let S be the restriction to $\mathcal{C}(\Omega)$ of the integration operator $\int_\Omega \varphi dm$, $\varphi \in \mathcal{B}(\Sigma)$. Then $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y^{**}))$. It remains to show that $(S\varphi)x \in Y$ for all $\varphi \in \mathcal{C}(\Omega)$ and $x \in X$. By (1),

$$\langle y^*, (S\varphi)x \rangle = \langle S\varphi, x \otimes y^* \rangle = \langle y^*, \int_\Omega \varphi dm_x \rangle, \quad y^* \in Y^*.$$

Hence, $(S\varphi)x \in Y^{**}$ is the composition of the weak*-to-weak* continuous map $y^* \mapsto \langle y^*, m_x(\cdot) \rangle$ from Y^* to $\mathcal{C}(\Omega)^*$ and the weak* continuous functional $\mu \mapsto \int_\Omega \varphi d\mu$ on $\mathcal{C}(\Omega)^*$. Therefore, $(S\varphi)x$ is a weak* continuous functional on Y^* , and $(S\varphi)x \in Y$ as desired.

For the “in this case” part, the first two claims come from Lemma 2.2. The third claim was proved in [20, Proposition 4.1]; for an alternative proof, see Corollary 2.7 below.

Finally, to show the weak*-countable additivity of m , let (E_n) be a sequence of pairwise disjoint members of Σ . Denote $f_k := \sum_{n=1}^k m(E_n)$, $k \in \mathbb{N}$, and $f := m(\bigcup_{n=1}^\infty E_n)$. By the countable additivity of μ_{x,y^*} , we have

$$\langle x \otimes y^*, f \rangle = \mu_{x,y^*}(\bigcup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \mu_{x,y^*}(E_n) = \lim_{k \rightarrow \infty} \langle x \otimes y^*, f_k \rangle$$

for all $x \in X$ and $y^* \in Y^*$. Since also the sequence (f_k) is bounded (in fact, $\|f_k\| \leq \|m\|(\Omega)$), $f_k \rightarrow f$ pointwise on $X \hat{\otimes}_\pi Y^*$. This means that

$$\langle u, m(\bigcup_{n=1}^\infty E_n) \rangle = \sum_{n=1}^\infty \langle u, m(E_n) \rangle$$

for all $u \in X \hat{\otimes}_\pi Y^*$, as desired. \square

In the classical case when $S \in \mathcal{L}(\mathcal{C}(\Omega), Y)$ and $m : \Sigma \rightarrow Y^{**}$ is its representing measure, the integration operator $\hat{S} \in \mathcal{L}(\mathcal{B}(\Sigma), Y^{**})$, $\hat{S}\varphi = \int_\Omega \varphi dm$, $\varphi \in \mathcal{B}(\Sigma)$, extends the operator S from $\mathcal{C}(\Omega)$ to $\mathcal{B}(\Sigma)$, where $\mathcal{C}(\Omega)$ sits as a closed subspace. And, in turn, $S^{**} \in \mathcal{L}(\mathcal{C}(\Omega)^{**}, Y^{**})$ extends the operator \hat{S} from $\mathcal{B}(\Sigma)$ to $\mathcal{C}(\Omega)^{**}$, where $\mathcal{B}(\Sigma)$ sits as a closed subspace.

In [20, the proof of Proposition 4.1], we pointed out that a similar phenomenon occurs also in the general case when $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. In the following, we shall make this precise.

Let $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ and let $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ be its representing measure. Then, as above, the integration operator $\hat{S} \in \mathcal{L}(\mathcal{B}(\Sigma), \mathcal{L}(X, Y^{**}))$ extends the operator S . More precisely, let

$$J : \mathcal{L}(X, Y) \rightarrow \mathcal{L}(X, Y^{**}), \quad J(A) = j_Y A, \quad A \in \mathcal{L}(X, Y),$$

be the natural isometric embedding. Then

$$JS = \hat{S}|_{\mathcal{C}(\Omega)}.$$

To understand in which sense S^{**} “extends” \hat{S} , recall that $\mathcal{L}(X, Y^{**}) = (X \hat{\otimes}_{\pi} Y^*)^*$ as Banach spaces (π denotes the projective tensor norm, as usual), and put

$$P := (j_{X \hat{\otimes}_{\pi} Y^*})^*.$$

Then P is the (natural) projection from $\mathcal{L}(X, Y^{**})^{**} = (X \hat{\otimes}_{\pi} Y^*)^{***}$ onto $\mathcal{L}(X, Y^{**}) = (X \hat{\otimes}_{\pi} Y^*)^*$.

Theorem 2.5. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Assume that $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ and let $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ be its representing measure. Then, with the above notation,*

$$(2) \quad \hat{S} = PJ^{**}S^{**}|_{\mathcal{B}(\Sigma)}.$$

Proof. It suffices to verify that

$$\hat{S}\chi_E = PJ^{**}S^{**}\chi_E \quad \text{for all } E \in \Sigma.$$

Then by linearity, (2) holds on $\mathfrak{S}(\Sigma)$, and by density, (2) holds on $\mathcal{B}(\Sigma)$.

For this end, in turn, it suffices to verify that

$$(3) \quad \langle x \otimes y^*, \hat{S}\chi_E \rangle = \langle x \otimes y^*, PJ^{**}S^{**}\chi_E \rangle, \quad x \in X, y^* \in Y^*.$$

For the left-hand side of (3), we have

$$(4) \quad \langle x \otimes y^*, \hat{S}\chi_E \rangle = \langle x \otimes y^*, m(E) \rangle = \langle y^*, m(E)x \rangle = \mu_{x, y^*}(E).$$

For the right-hand side of (3), we have, considering $\mathcal{C}(\Omega)^*$ embedded in $\mathcal{B}(\Sigma)^*$,

$$\begin{aligned} \langle x \otimes y^*, PJ^{**}S^{**}\chi_E \rangle &= \langle j_{X \hat{\otimes}_{\pi} Y^*}(x \otimes y^*), J^{**}S^{**}\chi_E \rangle \\ &= \langle \chi_E, S^*J^*j_{X \hat{\otimes}_{\pi} Y^*}(x \otimes y^*) \rangle = \langle \chi_E, S^*(x \otimes y^*) \rangle, \end{aligned}$$

because

$$\langle A, J^*j_{X \hat{\otimes}_{\pi} Y^*}(x \otimes y^*) \rangle = \langle x \otimes y^*, J(A) \rangle = \langle y^*, j_Y Ax \rangle = \langle Ax, y^* \rangle = \langle A, x \otimes y^* \rangle$$

for all $A \in \mathcal{L}(X, Y)$. But it is clear that

$$S^*(x \otimes y^*) = S_x^*y^*.$$

Indeed, for every $\varphi \in \mathcal{C}(\Omega)$, we have

$$\langle \varphi, S^*(x \otimes y^*) \rangle = \langle S\varphi, x \otimes y^* \rangle = \langle (S\varphi)x, y^* \rangle = \langle S_x\varphi, y^* \rangle = \langle \varphi, S_x^*y^* \rangle.$$

Therefore, using that $S_x^*y^* = \mu_{x, y^*}$ (see Lemma 2.2), we obtain

$$(5) \quad \langle x \otimes y^*, PJ^{**}S^{**}\chi_E \rangle = \langle \chi_E, \mu_{x, y^*} \rangle = \mu_{x, y^*}(E).$$

From (4) and (5), we get that (3) holds. \square

Remark 2.6. The above proof does not require the uniqueness of the representing measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ of $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ nor how the measure m is built.

Corollary 2.7 (see [20, Proposition 4.1]). *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Assume that $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ and let $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ be its representing measure. Then $\|m\|(\Omega) = \|S\|$.*

Proof. It is well known that $\|m\|(\Omega) = \|\hat{S}\|$ (see, e.g., [8, p. 6, Theorem 13]). But

$$\|S\| = \|JS\| = \|\hat{S}|_{\mathcal{C}(\Omega)}\| \leq \|\hat{S}\| = \|PJ^{**}S^{**}|_{\mathcal{B}(\Sigma)}\| \leq \|S\|.$$

□

Thanks to Theorem 2.5, we have an alternative proof for the uniqueness of the representing measure m .

Corollary 2.8 (see Proposition 2.3). *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Then the representing measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ of an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ is unique.*

Proof. Let $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ be a representing measure of $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. Then for all $E \in \Sigma$, we have

$$m(E) = \hat{S}\chi_E = PJ^{**}S^{**}\chi_E.$$

Hence, if $m_1, m_2 : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ are representing measures of S , then $m_1(E) = m_2(E)$ for all $E \in \Sigma$. □

For $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, together with its representing measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$, there also exists its classical representing measure, say $\mu : \Sigma \rightarrow \mathcal{L}(X, Y)^{**}$ (given by the Bartle–Dunford–Schwartz theorem). Let $\hat{\hat{S}} \in \mathcal{L}(\mathcal{B}(\Sigma), \mathcal{L}(X, Y)^{**})$ denote the corresponding integration operator, i.e., $\hat{\hat{S}} = \int_{\Omega} \varphi d\mu$, $\varphi \in \mathcal{B}(\Sigma)$. As is well known (this was also mentioned above), $S^{**}|_{\mathcal{B}(\Sigma)} = \hat{\hat{S}}$. Hence, Theorem 2.5 tells us that

$$\hat{S} = PJ^{**}\hat{\hat{S}}.$$

On characteristic functions, this gives the following formula (6) which connects the measures m and μ .

Corollary 2.9. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Assume that $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, and let $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ and $\mu : \Sigma \rightarrow \mathcal{L}(X, Y)^{**}$ be its representing measures. Then*

$$(6) \quad m(E) = PJ^{**}\mu(E) \quad \text{for all } E \in \Sigma.$$

Moreover, if S is weakly compact, then m takes its values in $\mathcal{L}(X, Y)$, and the measures m and μ coincide. In this case, the measure $m : \Sigma \rightarrow \mathcal{L}(X, Y)$ is countably additive and regular.

Proof. By the above, only the “moreover” part needs a proof. From the Bartle–Dunford–Schwartz theory [2] (see, e.g., [8, p. 153, Theorem 5]), it is well known that if $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ is weakly compact, then μ takes its values in $\mathcal{L}(X, Y)$ and $\mu : \Sigma \rightarrow \mathcal{L}(X, Y)$ is countably additive. It is also regular (see [8, p. 159, Corollary 14]). But for every $A \in \mathcal{L}(X, Y)$, considering $\mathcal{L}(X, Y)$ embedded in $\mathcal{L}(X, Y)^{**}$, we have

$$\begin{aligned} \langle x \otimes y^*, PJ^{**}(A) \rangle &= \langle j_{X \hat{\otimes}_\pi Y^*}(x \otimes y^*), J^{**}(A) \rangle \\ &= \langle A, J^* j_{X \hat{\otimes}_\pi Y^*}(x \otimes y^*) \rangle = \langle x \otimes y^*, j_Y A \rangle \end{aligned}$$

for all $x \in X$ and $y^* \in Y^*$, implying that $PJ^{**}(A) = j_Y A$. Therefore, by (6),

$$m(E) = PJ^{**}\mu(E) = j_Y \mu(E)$$

for all $E \in \Sigma$. This means that m takes its values in $\mathcal{L}(X, Y)$ and considering $\mathcal{L}(X, Y)$ embedded in $\mathcal{L}(X, Y^{**})$, the measures $m : \Sigma \rightarrow \mathcal{L}(X, Y)$ and $\mu : \Sigma \rightarrow \mathcal{L}(X, Y)$ coincide. \square

The next example shows that the fact that the representing measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ of an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ takes its values in $\mathcal{L}(X, Y)$ does not imply the weak compactness of the operator S .

Example 2.10. Denote by $\beta\mathbb{N}$ the Čech–Stone compactification of \mathbb{N} . As is well known, $\mathcal{C}(\beta\mathbb{N}) = \ell_\infty$. Consider the identity operator $I \in \mathcal{L}(\ell_\infty, \ell_\infty) = \mathcal{L}(\mathcal{C}(\beta\mathbb{N}), \mathcal{L}(\ell_1, \mathbb{K}))$. Since ℓ_∞ is not reflexive, I is a non-weakly compact operator. However, its representing measure $m : \Sigma \rightarrow \mathcal{L}(\ell_1, \mathbb{K}^{**})$ takes its values in $\mathcal{L}(\ell_1, \mathbb{K}) = \mathcal{L}(\ell_1, \mathbb{K}^{**})$.

Remark 2.11. Let $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ and let $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ be the representing measure of S . By definition of the measures m_x , the measure m takes its values in $\mathcal{L}(X, Y)$ if and only if all $m_x : \Sigma \rightarrow Y^{**}$, $x \in X$, take their values in Y . Since m_x is the representing measure of S_x , by the Bartle–Dunford–Schwartz theory (see, e.g., [8, p. 153, Theorem 5]), this is equivalent to the fact that all operators $S_x \in \mathcal{L}(\mathcal{C}(\Omega), Y)$, $x \in X$, are weakly compact. This clearly happens when S is weakly compact.

Remark 2.12. Let $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. In the above, we only needed (and used) the fact (from [20]) that a representing measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ exists for S . For completeness, let us recall how m is built in [20, Section

4]. Let $S_x \in \mathcal{L}(\mathcal{C}(\Omega), Y)$, $x \in X$, be defined (as above) by $S_x \varphi = (S\varphi)x$, $\varphi \in \mathcal{C}(\Omega)$, and let $m_x : \Sigma \rightarrow Y^{**}$ be its representing measure (given by the Bartle–Dunford–Schwartz theorem). Then $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ is defined by

$$\langle y^*, m(E)x \rangle = \langle y^*, m_x(E) \rangle, \quad E \in \Sigma,$$

for all $x \in X$ and $y^* \in Y^*$.

3. INTEGRATION OF p -CONTINUOUS VECTOR-VALUED FUNCTIONS WITH RESPECT TO AN OPERATOR-VALUED MEASURE

Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. The Banach space $\mathcal{C}_p(\Omega, X)$ of p -continuous X -valued functions [19] is formed by all $f \in \mathcal{C}(\Omega, X)$ such that $f(\Omega)$ is p -compact (i.e., there exists a sequence $(x_n) \in \ell_p(X)$ (or $(x_n) \in c_0(X)$ when $p = \infty$) such that $f(\Omega) \subset \{\sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_p}\}$). It follows from properties of p -compactness that $\mathcal{C}_p(\Omega, X) \subset \mathcal{C}_q(\Omega, X)$ if $p \leq q$, and $\mathcal{C}_\infty(\Omega, X) = \mathcal{C}(\Omega, X)$. The space $\mathcal{C}_p(\Omega, X)$ becomes a Banach space endowed with the norm

$$\|f\|_{\mathcal{C}_p(\Omega, X)} = \inf \|(x_n)\|_p,$$

where the infimum is taken over all sequences $(x_n) \in \ell_p(X)$ (or $(x_n) \in c_0(X)$ when $p = \infty$) such that $f(\Omega) \subset \{\sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_p}\}$, and $\mathcal{C}_\infty(\Omega, X) = \mathcal{C}(\Omega, X)$ as Banach spaces (see [19, Proposition 3.6]).

By Grothendieck's classics [16] (see, e.g., [22, pp. 49–50]), we know that

$$\mathcal{C}(\Omega, X) = \mathcal{C}(\Omega) \hat{\otimes}_\varepsilon X$$

as Banach spaces, where ε denotes the injective tensor norm, under the canonical isometric isomorphism $\varphi x \leftrightarrow \varphi \otimes x$, $\varphi \in \mathcal{C}(\Omega)$ and $x \in X$. One of the main results of [19] is that

$$\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega) \hat{\otimes}_{d_p} X$$

as Banach spaces, where d_p denotes the right Chevet–Saphar tensor norm (see [23] or, e.g., [22, Chapter 6] for the definition and properties; we do not need the definition in this paper).

Let $m : \Sigma \rightarrow \mathcal{L}(X, Y)$ be a vector measure. It is well known that the “algebraic” integral $\int_\Omega (\cdot) dm$ is defined on $\mathcal{S}(\Sigma, X)$. (The definition passes from vector-valued characteristic functions $\chi_E x$, $E \in \Sigma$, $x \in X$, to functions in $\mathcal{S}(\Sigma, X)$ by linearity.)

The classical Dinculeanu–Singer representation theorem requires the integration on $\mathcal{C}(\Omega, X)$. The corresponding integral was built by Dinculeanu (see [11, II.7.1, II.9.1, and p. 398, Theorem 9]; an early idea of this integral can be found in [15] and [1]). In fact, the Dinculeanu integral was built on

$\mathcal{B}(\Sigma, X)$, where $\mathcal{C}(\Omega, X)$ sits as a closed subspace, and then restricted to $\mathcal{C}(\Omega, X)$. On the other hand, the Dinculeanu integral restricted to $\mathcal{S}(\Sigma, X)$ coincides with the “algebraic” integral.

The existence of the Dinculeanu integral requires from m much more than does the existence of the elementary Bartle integral, where the semivariation $\|m\|(\Omega)$ was needed to be finite. Namely, a much bigger “semivariation” than $\|m\|(\Omega)$ must be finite. Let us call it the *Gowurin–Dinculeanu semivariation* (it was introduced by Gowurin [15] and deeply studied by Dinculeanu (see, e.g., [11, I.4])).

To be able to integrate on $\mathcal{C}_p(\Omega, X)$, we shall need an “intermediate semivariation”, depending on p , which, in the “limit” cases for $\mathcal{C}_1(\Omega, X)$ and $\mathcal{C}_\infty(\Omega, X) = \mathcal{C}(\Omega, X)$, coincides with the (usual) semivariation $\|m\|(\Omega)$ and the Gowurin–Dinculeanu semivariation, respectively (see Example 3.1 below).

Before introducing our “intermediate semivariation”, we shall need the description of the dual space $\mathcal{C}_p(\Omega, X)^*$ as a space of operators from $\mathcal{C}(\Omega)$ to X^* . Recall (see, e.g., [22, p. 142]) that the dual space operator ideal (we follow the terminology of [21]) of the Chevet–Saphar tensor norm d_p coincides with $\mathcal{P}_{p'}$, i.e., $(Z \hat{\otimes}_{d_p} X)^* = \mathcal{P}_{p'}(Z, X^*)$ as Banach spaces (here Z is an arbitrary Banach space). (Recall that $\mathcal{P}_q = (\mathcal{P}_q, \|\cdot\|_{\mathcal{P}_q})$, $1 \leq q \leq \infty$, denotes the Banach operator ideal of absolutely q -summing operators.) Since $\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega) \hat{\otimes}_{d_p} X$ as Banach spaces, we have

$$\mathcal{C}_p(\Omega, X)^* = \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*),$$

as Banach spaces, under the duality

$$\langle \varphi x, T \rangle = \langle x, T\varphi \rangle, \quad \varphi \in \mathcal{C}(\Omega), x \in X, T \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*).$$

Let $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ be a bounded vector measure. Notice that this clearly encompasses the seemingly more general case when m takes its values in $\mathcal{L}(X, Y)$, because Y is canonically embedded in Y^{**} . Then, for every $y^* \in Y^*$,

$$m_{y^*} := (m(\cdot))^* y^* : \Sigma \rightarrow X^*$$

is clearly a bounded vector measure. From the beginning of Section 2, we know that

$$\langle x, m_{y^*}(E) \rangle = \langle x, (m(E))^* y^* \rangle = \langle y^*, m(E)x \rangle = \mu_{x, y^*}(E),$$

and therefore, for all $\varphi \in \mathcal{B}(\Sigma)$,

$$(7) \quad \left\langle \int_{\Omega} \varphi dm, x \otimes y^* \right\rangle = \int_{\Omega} \varphi d\mu_{x, y^*} = \left\langle x, \int_{\Omega} \varphi dm_{y^*} \right\rangle.$$

Denote by I_{y^*} the restriction of the latter integral from $\mathcal{B}(\Sigma)$ to $\mathcal{C}(\Omega)$, i.e., for every $y^* \in Y^*$,

$$I_{y^*}\varphi = \int_{\Omega} \varphi dm_{y^*}, \quad \varphi \in \mathcal{C}(\Omega).$$

Then $I_{y^*} \in \mathcal{L}(\mathcal{C}(\Omega), X^*)$ and $m_{y^*} : \Sigma \rightarrow X^*$ is its representing measure.

Let $1 \leq q \leq \infty$. We define the q -semivariation $\|m\|_q(\Omega)$ of a bounded vector measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ by

$$\|m\|_q(\Omega) = \sup_{y^* \in B_{Y^*}} \|I_{y^*}\|_{\mathcal{P}_q}.$$

We say that a bounded vector measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ is of *bounded q -semivariation* if $\|m\|_q(\Omega) < \infty$. It follows from the inclusion theorem for absolutely q -summing operators (see, e.g., [7, p. 39, Theorem 2.8]) that

$$\|m\|_{\infty}(\Omega) \leq \|m\|_q(\Omega) \leq \|m\|_p(\Omega) \leq \|m\|_1(\Omega) \quad \text{if } 1 \leq p \leq q \leq \infty.$$

Example 3.1. Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ be a bounded vector measure. Then $\|m\|_{\infty}(\Omega) = \|m\|(\Omega)$, the semivariation of m , and $\|m\|_1(\Omega)$ coincides with the Gowurin–Dinculeanu semivariation.

Proof. Let $y^* \in Y^*$. Since $(\mathcal{P}_{\infty}, \|\cdot\|_{\mathcal{P}_{\infty}}) = (\mathcal{L}, \|\cdot\|)$, we have that $\|I_{y^*}\|_{\mathcal{P}_{\infty}} = \|I_{y^*}\|$. And since m_{y^*} is the representing measure of $I_{y^*} \in \mathcal{L}(\mathcal{C}(\Omega), X^*)$, we have that $\|I_{y^*}\| = \|m_{y^*}\|(\Omega)$, by the Bartle–Dunford–Schwartz theorem. Therefore

$$\begin{aligned} \|m\|_{\infty}(\Omega) &= \sup_{y^* \in B_{Y^*}} \|m_{y^*}\|(\Omega) \\ &= \sup \left\{ \left\| \sum_{E_i \in \Pi} \varepsilon_i m_{y^*}(E_i) \right\| : y^* \in B_{Y^*}, |\varepsilon_i| \leq 1, \Pi \right\} \\ &= \sup \left\{ \left| \left\langle x, \sum_{E_i \in \Pi} \varepsilon_i m_{y^*}(E_i) \right\rangle \right| : x \in B_X, y^* \in B_{Y^*}, |\varepsilon_i| \leq 1, \Pi \right\} \\ &= \sup \left\{ \left| \left\langle y^*, \left(\sum_{E_i \in \Pi} \varepsilon_i m(E_i) \right) x \right\rangle \right| : x \in B_X, y^* \in B_{Y^*}, |\varepsilon_i| \leq 1, \Pi \right\} \\ &= \sup \left\{ \left\| \left(\sum_{E_i \in \Pi} \varepsilon_i m(E_i) \right) x \right\| : x \in B_X, |\varepsilon_i| \leq 1, \Pi \right\} \\ &= \sup \left\{ \left\| \sum_{E_i \in \Pi} \varepsilon_i m(E_i) \right\| : |\varepsilon_i| \leq 1, \Pi \right\} = \|m\|(\Omega). \end{aligned}$$

We know that $\mathcal{P}_1(\mathcal{C}(\Omega), X^*) = \mathcal{C}_{\infty}(\Omega, X)^* = \mathcal{C}(\Omega, X)^*$. We also know that $I_{y^*} \in \mathcal{L}(\mathcal{C}(\Omega), X^*)$ is absolutely summing, i.e., $I_{y^*} \in \mathcal{P}_1(\mathcal{C}(\Omega), X^*)$ if

and only if its representing measure m_{y^*} is of bounded variation, and in this case, $\|I_{y^*}\|_{\mathcal{P}_1} = |m_{y^*}|(\Omega)$ (see, e.g., [8, p. 162, Theorem 3]). Hence

$$(8) \quad \|m\|_1(\Omega) = \sup_{y^* \in B_{Y^*}} |m_{y^*}|(\Omega),$$

which, thanks to [11, p. 55, Proposition 5], coincides with the Gowurin–Dinculeanu semivariation of m . Let us recall that in [8, p. 181], formula (8) is taken as the definition of the Gowurin–Dinculeanu semivariation of m . \square

Below, we shall need the following result which, among others, may be used for calculating $\|m\|_q(\Omega)$. For $y^* \in Y^*$, let

$$\hat{I}_{y^*} := \int_{\Omega} (\cdot) dm_{y^*} \in \mathcal{L}(\mathcal{B}(\Sigma), X^*)$$

denote the integration operator with respect to m_{y^*} .

Proposition 3.2. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \leq q \leq \infty$. Assume that $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then*

$$\|\hat{I}_{y^*}\|_{\mathcal{P}_q} = \|I_{y^*}\|_{\mathcal{P}_q} \quad \text{for all } y^* \in Y^*.$$

Proof. Since $\mathcal{C}(\Omega) \subset \mathcal{B}(\Sigma) \subset \mathcal{C}(\Omega)^{**}$ as closed subspaces, \hat{I}_{y^*} is an extension of I_{y^*} , and $(I_{y^*})^{**}$ is an extension of \hat{I}_{y^*} , we have that

$$\|I_{y^*}\|_{\mathcal{P}_q} \leq \|\hat{I}_{y^*}\|_{\mathcal{P}_q} \leq \|(I_{y^*})^{**}\|_{\mathcal{P}_q}.$$

Hence, if $\|I_{y^*}\|_{\mathcal{P}_q} = \infty$, then also $\|\hat{I}_{y^*}\|_{\mathcal{P}_q} = \infty$. If $\|I_{y^*}\|_{\mathcal{P}_q} < \infty$, i.e., I_{y^*} is absolutely q -summing, then also $(I_{y^*})^{**}$ is, and in this case, $\|(I_{y^*})^{**}\|_{\mathcal{P}_q} = \|I_{y^*}\|_{\mathcal{P}_q}$ (see, e.g., [7, p. 50, Proposition 2.19]). Therefore $\|I_{y^*}\|_{\mathcal{P}_q} = \|\hat{I}_{y^*}\|_{\mathcal{P}_q}$, as desired. \square

It is well known (see, e.g., [22, p. 11]) that $\mathcal{B}(\Sigma) \otimes X \subset \mathcal{B}(\Sigma, X)$ as a linear subspace, under the algebraic identification $\varphi \otimes x \leftrightarrow \varphi x$. This is used in the following result.

Theorem 3.3. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. Assume that $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then the formula*

$$(9) \quad \int_{\Omega} (\varphi x) dm = \left(\int_{\Omega} \varphi dm \right) x, \quad \varphi \in \mathcal{B}(\Sigma), x \in X,$$

*defines an integral on $\mathcal{B}(\Sigma) \hat{\otimes}_{d_p} X$ with respect to m if and only if $\|m\|_{p'}(\Omega) < \infty$. In this case, the integration operator \hat{U} belongs to $\mathcal{L}(\mathcal{B}(\Sigma) \hat{\otimes}_{d_p} X, Y^{**})$, $\|\hat{U}\| = \|m\|_{p'}(\Omega)$, the restriction of \hat{U} to $\mathcal{S}(\Sigma, X) = \mathcal{S}(\Sigma) \otimes X$ coincides with the “algebraic” integral, and $\hat{U}^* y^* = \hat{I}_{y^*}$ for all $y^* \in Y^*$.*

Moreover, the measure m takes its values in $\mathcal{L}(X, Y)$ if and only if the integration operator \hat{U} takes its values in Y .

Proof. First of all, notice that if the main part of the theorem holds true, then the “if” part of the “moreover” part is clear from (9). Indeed, assume that $\text{ran } \hat{U} \subset Y$. Since

$$m(E) = \int_{\Omega} \chi_E dm \quad \text{for all } E \in \Sigma,$$

by (9), we have that

$$m(E)x = \int_{\Omega} (\chi_E x) dm = \hat{U}(\chi_E \otimes x) \in Y \quad \text{for all } E \in \Sigma \text{ and } x \in X.$$

This means that $\text{ran } m \subset \mathcal{L}(X, Y)$.

To prove the theorem and to encompass also the “only if” part of the “moreover” part, let $W := Y^{**}$ or $W := Y$.

On the right-hand side of (9), the integral is just the elementary Bartle integral with respect to m . Denote by $\hat{S} \in \mathcal{L}(\mathcal{B}(\Sigma), \mathcal{L}(X, W))$ this integration operator. Since, as is well known, $\mathcal{L}(Z, \mathcal{L}(X, W))$ is canonically isometrically isomorphic to $\mathcal{L}(Z \otimes_{\pi} X, W) = \mathcal{L}(Z \hat{\otimes}_{\pi} X, W)$ (for any Banach spaces X, W , and Z), there exists a unique linear operator $\hat{U} : \mathcal{B}(\Sigma) \otimes X \rightarrow W$ such that

$$\hat{U}(\varphi \otimes x) = (\hat{S}\varphi)x, \quad \varphi \in \mathcal{B}(\Sigma), x \in X,$$

and $\hat{U} \in \mathcal{L}(\mathcal{B}(\Sigma) \otimes_{\pi} X, W)$. Hence, by (9),

$$\int_{\Omega} (\varphi x) dm = \hat{U}(\varphi \otimes x), \quad \varphi \in \mathcal{B}(\Sigma), x \in X,$$

giving that

$$\int_{\Omega} f dm = \hat{U}f, \quad f \in \mathcal{S}(\Sigma, X) = \mathcal{S}(\Sigma) \otimes X.$$

It remains to prove that

$$(10) \quad \nu := \sup\{\|\hat{U}v\| : v \in \mathcal{B}(\Sigma) \otimes X, \|v\|_{d_p} \leq 1\} = \|m\|_{p'}(\Omega).$$

Then, in the case when $\|m\|_{p'}(\Omega) < \infty$ or, equivalently, $\hat{U} \in \mathcal{L}(\mathcal{B}(\Sigma) \otimes_{d_p} X, W)$, by passing to the unique continuous linear extension of \hat{U} , we get that $\hat{U} \in \mathcal{L}(\mathcal{B}(\Sigma) \hat{\otimes}_{d_p} X, W)$ and $\|\hat{U}\| = \|m\|_{p'}(\Omega)$. Therefore, the integral $\int_{\Omega}(\cdot) dm$ is defined on $\mathcal{B}(\Sigma) \hat{\otimes}_{d_p} X$ by

$$\int_{\Omega} v dm = \hat{U}v, \quad v \in \mathcal{B}(\Sigma) \hat{\otimes}_{d_p} X.$$

Let us now prove equality (10). Fix an arbitrary $y^* \in Y^*$. Then, for all $\varphi \in \mathcal{B}(\Sigma)$ and $x \in X$, by (7), we have

$$(11) \quad \begin{aligned} \langle x, \hat{I}_{y^*} \varphi \rangle &= \langle \hat{S}\varphi, x \otimes y^* \rangle = \langle y^*, (\hat{S}\varphi)x \rangle \\ &= \langle y^*, \hat{U}(\varphi \otimes x) \rangle = \langle \varphi \otimes x, \hat{U}^* y^* \rangle = \langle x, (\hat{U}^* y^*) \varphi \rangle; \end{aligned}$$

for the two last equalities, recall that we have

$$\hat{U}^* \in \mathcal{L}(W^*, (\mathcal{B}(\Sigma) \otimes_\pi X)^*),$$

so that $\hat{U}^* y^* \in (\mathcal{B}(\Sigma) \otimes_\pi X)^* = \mathcal{L}(\mathcal{B}(\Sigma), X^*)$. Therefore, $\hat{I}_{y^*} = \hat{U}^* y^*$ and thus

$$\begin{aligned} \|\hat{I}_{y^*}\|_{\mathcal{P}_{p'}} &= \|\hat{U}^* y^*\|_{\mathcal{P}_{p'}} = \sup\{|\langle v, \hat{U}^* y^* \rangle| : v \in \mathcal{B}(\Sigma) \otimes X, \|v\|_{d_p} \leq 1\} \\ &= \sup\{|\langle y^*, \hat{U} v \rangle| : v \in \mathcal{B}(\Sigma) \otimes X, \|v\|_{d_p} \leq 1\} \leq \nu \|y^*\|. \end{aligned}$$

Hence,

$$\nu \geq \|m\|_{p'}(\Omega).$$

For the reverse inequality, let $v = \sum_{i=1}^n \varphi_i \otimes x_i \in \mathcal{B}(\Sigma) \otimes_{d_p} X$. For any $y^* \in Y^*$, by (11), we have

$$\begin{aligned} |\langle y^*, \hat{U} v \rangle| &= \left| \langle y^*, \sum_{i=1}^n \hat{U}(\varphi_i \otimes x_i) \rangle \right| = \left| \sum_{i=1}^n \langle y^*, \hat{U}(\varphi_i \otimes x_i) \rangle \right| \\ &= \left| \sum_{i=1}^n \langle x_i, \hat{I}_{y^*} \varphi_i \rangle \right| \leq \|(x_i)_{i=1}^n\|_p \|(\hat{I}_{y^*} \varphi_i)_{i=1}^n\|_{p'} \leq \|(x_i)_{i=1}^n\|_p \|\hat{I}_{y^*}\|_{\mathcal{P}_{p'}} \|(\varphi_i)_{i=1}^n\|_{p'}^w. \end{aligned}$$

Taking first the infimum over all the representations of $v \in \mathcal{B}(\Sigma) \otimes_{d_p} X$ and then the supremum over $y^* \in B_{Y^*}$, by Proposition 3.2, we obtain that

$$\|\hat{U} v\| \leq \|m\|_{p'}(\Omega) \|v\|_{d_p},$$

hence,

$$\nu \leq \|m\|_{p'}(\Omega),$$

and (10) holds.

Finally, if $\hat{U} \in \mathcal{L}(\mathcal{B}(\Sigma) \hat{\otimes}_{d_p} X, W)$, then we have

$$\hat{U}^* \in \mathcal{L}(W^*, (\mathcal{B}(\Sigma) \hat{\otimes}_{d_p} X)^*) = \mathcal{L}(W^*, \mathcal{P}_{p'}(\mathcal{B}(\Sigma), X^*)),$$

and equalities (11) hold true, giving that $\hat{I}_{y^*} = \hat{U}^* y^*$ for all $y^* \in Y^*$. \square

As we mentioned in the beginning of this section, $\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega) \hat{\otimes}_{d_p} X$ as Banach spaces, under the identification $\varphi x \leftrightarrow \varphi \otimes x$. On the other hand, let us observe that $\mathcal{C}(\Omega) \hat{\otimes}_{d_p} X$ is a closed subspace of $\mathcal{B}(\Sigma) \hat{\otimes}_{d_p} X$. Indeed, it is well known that $\mathcal{C}(\Omega)^*$ is isometrically isomorphic to an $L_1(\mu)$ -space for some measure μ , i.e., $\mathcal{C}(\Omega)$ is an L_1 -predual space. Thanks to Fakhoury [13, Corollary 3.3] and Grothendieck [17, Theorem 1] (see, e.g., [6, pp. 76, 81]), L_1 -predual spaces are ideals in their “superspaces” (for more details, see [18, p. 49]). In particular, $\mathcal{C}(\Omega)$ is an ideal in $\mathcal{B}(\Sigma)$. But then (see [21, Proposition 2.4]) $\mathcal{C}(\Omega) \otimes_{d_p} X$ is a subspace of $\mathcal{B}(\Sigma) \hat{\otimes}_{d_p} X$, and therefore $\mathcal{C}(\Omega) \hat{\otimes}_{d_p} X = \overline{\mathcal{C}(\Omega) \otimes_{d_p} X}$ is a closed subspace of $\mathcal{B}(\Sigma) \hat{\otimes}_{d_p} X$.

Therefore $\mathcal{C}_p(\Omega, X)$ is a closed subspace of $\mathcal{B}(\Sigma) \hat{\otimes}_{d_p} X$, and Theorem 3.3 almost immediately yields the integration result below (Theorem 3.4).

Let $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ be a vector measure of bounded p' -semivariation. Denote by

$$U := \hat{U}|_{\mathcal{C}_p(\Omega, X)} = \hat{U}|_{\mathcal{C}(\Omega) \hat{\otimes}_{d_p} X}$$

the restriction to $\mathcal{C}_p(\Omega, X)$ of the integration operator \hat{U} given by Theorem 3.3, i.e.,

$$Uf = \int_{\Omega} f \, dm, \quad f \in \mathcal{C}_p(\Omega, X).$$

Theorem 3.4. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. Assume that $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ is a vector measure of bounded p' -semivariation. Then the formula (9) defines an integral on $\mathcal{C}_p(\Omega, X)$ with respect to m , the integration operator U belongs to $\mathcal{L}(\mathcal{C}_p(\Omega, X), Y^{**})$, $\|U\| = \|m\|_{p'}(\Omega)$, and $U^*y^* = I_{y^*}$ for all $y^* \in Y^*$.*

Moreover, if the integration operator U takes its values in Y , in particular, this is the case when the measure m takes its values in $\mathcal{L}(X, Y)$, then

$$U^{**}(\chi_E \otimes x) = m(E)x \quad \text{for all } E \in \Sigma \text{ and } x \in X,$$

where $\chi_E \otimes x \in \mathcal{C}_p(\Omega, X)^{**}$ is defined in the canonical way:

$$\langle A, \chi_E \otimes x \rangle = \langle A^*x, \chi_E \rangle, \quad A \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*) = \mathcal{C}_p(\Omega, X)^*.$$

Proof. For the main part of the theorem, in view of Theorem 3.3, we only need to show that $\|U\| \geq \|m\|_{p'}(\Omega)$ (because $\|U\| \leq \|\hat{U}\| = \|m\|_{p'}(\Omega)$) and $U^*y^* = I_{y^*}$ for all $y^* \in Y^*$.

Let $y^* \in Y^*$. Using that $U \in \mathcal{L}(\mathcal{C}(\Omega) \hat{\otimes}_{d_p} X, Y^{**})$ and $(\mathcal{C}(\Omega) \hat{\otimes}_{d_p} X)^* = \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$, so that $U^* \in \mathcal{L}(Y^{***}, \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*))$, we get from (11) that

$$\langle x, I_{y^*}\varphi \rangle = \langle y^*, U(\varphi \otimes x) \rangle = \langle \varphi \otimes x, U^*y^* \rangle = \langle x, (U^*y^*)\varphi \rangle$$

for all $x \in X$ and $\varphi \in \mathcal{C}(\Omega)$. Therefore $I_{y^*} = U^*y^*$ and

$$\|I_{y^*}\|_{\mathcal{P}_{p'}} = \|U^*y^*\|_{\mathcal{P}_{p'}} \leq \|U^*\| \|y^*\| = \|U\| \|y^*\|$$

for all $y^* \in Y^*$. This yields that

$$\|m\|_{p'}(\Omega) = \sup_{y^* \in B_{Y^*}} \|I_{y^*}\|_{\mathcal{P}_{p'}} \leq \|U\|.$$

Now, for the “moreover” part, assume that $\text{ran } U \subset Y$. Then $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$. Let $E \in \Sigma$, $x \in X$, and $y^* \in Y^*$. Then,

$$\begin{aligned} \langle y^*, U^{**}(\chi_E \otimes x) \rangle &= \langle U^*y^*, \chi_E \otimes x \rangle = \langle I_{y^*}, \chi_E \otimes x \rangle = \langle (I_{y^*})^*x, \chi_E \rangle \\ &= \langle x, (I_{y^*})^{**}\chi_E \rangle = \langle x, \hat{I}_{y^*}\chi_E \rangle = \langle y^*, \hat{U}(\chi_E \otimes x) \rangle, \end{aligned}$$

where the last equality holds by (11). Therefore, $U^{**}(\chi_E \otimes x) = \hat{U}(\chi_E \otimes x)$ for all $E \in \Sigma$ and $x \in X$. But, by (9), we have that

$$\hat{U}(\chi_E \otimes x) = \left(\int_{\Omega} \chi_E dm \right) x = m(E)x,$$

proving that $U^{**}(\chi_E \otimes x) = m(E)x$ for all $E \in \Sigma$ and $x \in X$.

Finally, let us recall from Theorem 3.3 that $\text{ran } m \subset \mathcal{L}(X, Y)$ if and only if $\text{ran } \hat{U} \subset Y$. Hence, in this case, $\text{ran } U \subset Y$. \square

Remark 3.5. From Example 3.1 and Theorem 3.4, it is clear that, in the special case when $p = \infty$, our integral coincides with the Dinculeanu integral from [11].

Remark 3.6. Our notion of the q -semivariation is different from the notion “ q -semivariation” introduced in Dinculeanu’s book [11, p. 246]. Let us call the latter “the Dinculeanu q -semivariation”. Its definition is as follows.

Let $1 \leq q \leq \infty$ and let $\mu : \Sigma \rightarrow \mathbb{R}$ be a positive finite measure; we may assume that $\mu(\Omega) = 1$. For a vector measure $m : \Sigma \rightarrow \mathcal{L}(X, Y)$, the *Dinculeanu q -semivariation* on Ω (see [11, p. 246]) is defined by

$$\tilde{m}_q(\Omega) = \sup \left\{ \left\| \sum_{E_i \in \Pi} m(E_i)x_i \right\| \right\},$$

where the supremum is taken over all finite partitions $\Pi = (E_i)_{i=1}^n$ of Ω and all finite systems $(x_i)_{i=1}^n \subset X$ such that $\left\| \sum_{i=1}^n \chi_{E_i} x_i \right\|_{L_{q'}(\mu, X)} \leq 1$, $n \in \mathbb{N}$. This notion is used in [11, II.13] to obtain the integral representation of an operator $U \in \mathcal{L}(L_p(\mu, X), Y)$, $1 \leq p < \infty$, with respect to a vector measure $m : \Sigma \rightarrow \mathcal{L}(X, Y)$ such that $\tilde{m}_{p'}(\Omega) < \infty$.

It can be easily verified that $\|m\|_1(\Omega) \leq \tilde{m}_1(\Omega)$ and $\|m\|_1(\Omega) = \tilde{m}_1(\Omega)$ if m is absolutely continuous with respect to μ (see [11, p. 246]). Since also $\tilde{m}_1(\Omega) \leq \tilde{m}_q(\Omega)$ (see [11, p. 247]), we have that

$$\|m\|_q(\Omega) \leq \|m\|_1(\Omega) \leq \tilde{m}_1(\Omega) \leq \tilde{m}_q(\Omega).$$

4. REPRESENTING MEASURE OF $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$

Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. Basing on Theorem 3.4, we may give the following definition whose special case when $p = \infty$, thanks to Example 3.1, coincides with the classical one, known from the Dinculeanu–Singer theorem.

Definition 4.1. Let $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$. A *representing measure* of U is a vector measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ of bounded p' -semivariation which

satisfies

$$(12) \quad Uf = \int_{\Omega} f \, dm \quad \text{for all } f \in \mathcal{C}_p(\Omega, X).$$

Remark 4.2. In the classical case of $\mathcal{C}(\Omega, X) = \mathcal{C}_{\infty}(\Omega, X)$, Definition 4.1 differs from the definition of representing measure by Brooks and Lewis [5, Definition 2.9]. Namely, we do not require that the measures $m_{y^*} : \Sigma \rightarrow X^*$, $y^* \in Y^*$ (see Section 3) were regular. They have this regularity property thanks to Theorem 4.8 below. More precisely, the regularity holds whenever $p \neq 1$ (and this condition is essential by Example 4.9).

Theorem 4.3. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. Assume that $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$, and let $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ be the representing measure of the associated operator $U^{\#} \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. Then m is a representing measure of U , $I_{y^*} = U^* y^* \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$ for all $y^* \in Y^*$, $\|U\| = \|m\|_{p'}(\Omega)$, and*

$$U^{**}(\chi_E \otimes x) = m(E)x \quad \text{for all } E \in \Sigma \text{ and } x \in X.$$

Proof. We know that $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ is a bounded vector measure. For all $\varphi \in \mathcal{C}(\Omega)$, $x \in X$, and $y^* \in Y^*$, by (7), we have that

$$\begin{aligned} \langle x, I_{y^*} \varphi \rangle &= \left\langle \int_{\Omega} \varphi \, dm, x \otimes y^* \right\rangle = \langle U^{\#} \varphi, x \otimes y^* \rangle = \langle y^*, (U^{\#} \varphi)x \rangle \\ &= \langle y^*, U(\varphi \otimes x) \rangle = \langle U^* y^*, \varphi \otimes x \rangle = \langle x, (U^* y^*) \varphi \rangle. \end{aligned}$$

Hence $I_{y^*} = U^* y^*$ for all $y^* \in Y^*$. Therefore $I_{y^*} \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$ and

$$\|m\|_{p'}(\Omega) = \sup_{y^* \in B_{Y^*}} \|U^* y^*\|_{\mathcal{P}_{p'}} = \|U^*\| = \|U\| < \infty.$$

Since m is of bounded p' -semivariation, the formula (9) defines

$$\int_{\Omega} (\cdot) \, dm \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y^{**})$$

(see Theorem 3.4). We only need to show (12), because then also the last claim holds true thanks to Theorem 3.4.

Let $\varphi \in \mathcal{C}(\Omega)$ and $x \in X$. Then

$$U(\varphi x) = (U^{\#} \varphi)x = \left(\int_{\Omega} \varphi \, dm \right) x = \int_{\Omega} (\varphi x) \, dm$$

by (9). It is well known (see, e.g., [22, p. 11]) that $\mathcal{C}(\Omega) \otimes X \subset \mathcal{C}_p(\Omega, X)$ as a linear subspace (under the algebraic identification $\varphi \otimes x \leftrightarrow \varphi x$ that was used in Section 3). Therefore, by linearity, (12) holds for every $f \in \mathcal{C}(\Omega) \otimes X$. If now $f \in \mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega) \hat{\otimes}_{d_p} X$ is arbitrary, then $f = \lim_n f_n$ in $\mathcal{C}_p(\Omega, X)$ for some $f_n \in \mathcal{C}(\Omega) \otimes X$. Hence,

$$Uf = \lim_n Uf_n = \lim_n \int_{\Omega} f_n \, dm \text{ in } Y.$$

On the other hand, by the definition of the integral,

$$\int_{\Omega} f \, dm = \lim_n \int_{\Omega} f_n \, dm \text{ in } Y^{**}.$$

Consequently, (12) holds. \square

Theorem 4.3 shows that a representing measure of $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ may be defined as the representing measure of its associated operator $U^{\#}$. Now we see that this is, in fact, the unique way to define a representing measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ for U .

Proposition 4.4. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. Assume that $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ is a representing measure of $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$. Then m is the representing measure of $U^{\#}$.*

Proof. Let $\varphi \in \mathcal{C}(\Omega)$ and $x \in X$. Then, using (9), we have

$$(U^{\#}\varphi)x = U(\varphi x) = \int_{\Omega} (\varphi x) \, dm = \left(\int_{\Omega} \varphi \, dm \right) x.$$

Thus

$$U^{\#}\varphi = \int_{\Omega} \varphi \, dm, \quad \varphi \in \mathcal{C}(\Omega).$$

\square

Since the representing measure of $U^{\#}$ is unique (see Proposition 2.3), the following is immediate from Proposition 4.4.

Corollary 4.5. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. Then the representing measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ of an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ is unique.*

In view of Proposition 4.4 and Corollary 2.9, the next corollary is immediate; the operators J and P were introduced before Theorem 2.5.

Corollary 4.6. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. Assume that $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$, let $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ be its representing measure, and let $\mu : \Sigma \rightarrow \mathcal{L}(X, Y)^{**}$ be the (classical) representing measure of the associated operator $U^{\#} \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. Then $m(E) = PJ^{**}\mu(E)$ for all $E \in \Sigma$.*

Moreover, if $U^{\#}$ is weakly compact, then m takes its values in $\mathcal{L}(X, Y)$, and the measures m and μ coincide. In this case, the measure $m : \Sigma \rightarrow \mathcal{L}(X, Y)$ is countably additive and regular.

Remark 4.7. Concerning the classical case of $\mathcal{C}(\Omega, X) = \mathcal{C}_\infty(\Omega, X)$, Corollary 4.6 provides, for the first time in the literature, a general formula connecting the representing measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ of $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ and the classical representing measure $\mu : \Sigma \rightarrow \mathcal{L}(X, Y)^{**}$ of $U^\# \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. In this sense, let us point out the partial result due to Dinculeanu (see [10, Theorems 4 and 5], or, e.g., [11, p. 388, Theorem 4]): if $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ and $U^\# \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ are *dominated* operators, then m takes its values in $\mathcal{L}(X, Y)$, and the measures m and μ coincide.

Recall that a vector measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ is *weakly regular* if $m_{y^*} : \Sigma \rightarrow X^*$ is regular for all $y^* \in Y^*$ (see, e.g., [8, p. 181]). The following result extends the classical Dinculeanu–Singer theorem (see, e.g., [8, p. 182]) in all its aspects (see Corollary 4.10 and the paragraph preceding it).

Theorem 4.8. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$.*

(a) *Every operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ has a unique representing measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$. This measure coincides with the representing measure of its associated operator $U^\# \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$.*

(b) *Assume that $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then, there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that m is its representing measure if and only if for all $y^* \in Y^*$,*

$$I_{y^*} \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*),$$

and the map $Y^ \rightarrow \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*) = \mathcal{C}_p(\Omega, X)^*$, $y^* \mapsto I_{y^*}$, is linear, bounded, and weak*-to-weak* continuous.*

In this case, $I_{y^} = U^*y^*$ for all $y^* \in Y^*$, $U^{**}(\chi_E \otimes x) = m(E)x$ for all $E \in \Sigma$ and $x \in X$, $\|U\| = \|m\|_{p'}(\Omega)$, $\|U^\#\| = \|m\|(\Omega)$, and m is a weakly regular measure if $p > 1$.*

Proof. (a) A representing measure for an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ (the associated operator $U^\#$ is of this type) always exists (see Remark 2.12). Theorem 4.3 and Proposition 4.4 show that the representing measures of U and $U^\#$ coincide. The measure is unique by Corollary 4.5.

(b) Let $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ be the representing measure of $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$. Then, by (a) and Theorem 4.3, m is also the representing measure of $U^\# \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, $I_{y^*} = U^*y^* \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$ for all $y^* \in Y^*$, $U^{**}(\chi_E \otimes x) = m(E)x$ for all $E \in \Sigma$ and $x \in X$, and $\|U\| = \|m\|_{p'}(\Omega)$. In particular, $U^* : y^* \mapsto I_{y^*}$ is linear, bounded, and weak*-to-weak* continuous. This shows the “only if” part.

For the “if” part, let $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ be a bounded vector measure. Denote by V the map given by the assumption, i.e.,

$$V : Y^* \rightarrow \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*) = \mathcal{C}_p(\Omega, X)^*, \quad y^* \mapsto I_{y^*}.$$

Since V is weak*-to-weak* continuous, there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^* = V$.

We only need to show that

$$\int_{\Omega} \varphi \, dm = U^{\#} \varphi, \quad \varphi \in \mathcal{C}(\Omega),$$

because then $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ is the representing measure of $U^{\#} \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, hence also the representing measure of U (see Theorem 4.3).

For every $\varphi \in \mathcal{C}(\Omega)$, $x \in X$, and $y^* \in Y^*$, using (7), we have that

$$\begin{aligned} \langle y^*, \left(\int_{\Omega} \varphi \, dm \right) x \rangle &= \left\langle \int_{\Omega} \varphi \, dm, x \otimes y^* \right\rangle = \left\langle x, \int_{\Omega} \varphi \, dm_{y^*} \right\rangle \\ &= \langle x, I_{y^*} \varphi \rangle = \langle x, (V y^*) \varphi \rangle = \langle \varphi x, V y^* \rangle \\ &= \langle \varphi x, U^* y^* \rangle = \langle U(\varphi x), y^* \rangle = \langle (U^{\#} \varphi) x, y^* \rangle. \end{aligned}$$

This proves that m is the representing measure of $U^{\#}$, as desired.

For the “in this case” part, the first three claims were already observed above. Since m is also the representing measure of $U^{\#} \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, by Corollary 2.7, we have that $\|U^{\#}\| = \|m\|(\Omega)$. Concerning the remaining claim about the weak regularity, recall that m_{y^*} is the representing measure of $I_{y^*} \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$ for every $y^* \in Y^*$. If $p > 1$, then $p' < \infty$, and I_{y^*} is a weakly compact operator (see, e.g., [7, p. 50, Theorem 2.17]). Therefore, m_{y^*} is regular (see, e.g., [8, p. 159, Corollary 14]) for all $y^* \in Y^*$. \square

The next example shows that, for $p = 1$, the measure m in Theorem 4.8 is not weakly regular in general.

Example 4.9. Let X be a Banach space and let Ω be a compact Hausdorff space such that there exists a non-weakly compact operator $S \in \mathcal{L}(\mathcal{C}(\Omega), X^*)$. Then, there exists an operator $U \in \mathcal{L}(\mathcal{C}_1(\Omega, X), \mathbb{K})$ such that its representing measure $m : \Sigma \rightarrow \mathcal{L}(X, \mathbb{K}) = X^*$ is not weakly regular.

Proof. Let $S \in \mathcal{L}(\mathcal{C}(\Omega), X^*)$ be a non-weakly compact operator. Then its representing measure $m : \Sigma \rightarrow X^{***}$ is not regular (see, e.g., [8, p. 159, Corollary 14]).

Since, as is well known, $\mathcal{L}(\mathcal{C}(\Omega), X^*)$ is canonically isometrically isomorphic to $(\mathcal{C}(\Omega) \otimes_{\pi} X)^* = (\mathcal{C}(\Omega) \hat{\otimes}_{\pi} X)^*$ and $\pi = d_1$, there exists a unique operator $U \in \mathcal{L}(\mathcal{C}(\Omega) \hat{\otimes}_{d_1} X, \mathbb{K}) = \mathcal{L}(\mathcal{C}_1(\Omega, X), \mathbb{K})$ such that $S = U^{\#}$. Then

m is the representing measure of U because m is the representing measure of $U^\# = S$ (see Theorem 4.3). We know that $U^* \in \mathcal{L}(\mathbb{K}^*, \mathcal{C}_1(\Omega, X)^*) = \mathcal{L}(\mathbb{K}, \mathcal{L}(\mathcal{C}(\Omega), X^*))$ and, by Theorem 4.3, $I_1 = U^*1$. On the other hand, for every $\varphi \in \mathcal{C}(\Omega)$ and $x \in X$, we have

$$((U^*1)\varphi)x = \langle \varphi x, U^*1 \rangle = 1 U(\varphi x) = (U^\# \varphi)x = (S\varphi)x,$$

meaning that $U^*1 = S$. Therefore, $I_1 = S$, and its representing measure, which is m , is not regular. Hence, the representing measure of U is not weakly regular. \square

Since $\mathcal{C}_\infty(\Omega, X) = \mathcal{C}(\Omega, X)$ and, hence, $\mathcal{P}_1(\mathcal{C}(\Omega), X^*) = \mathcal{C}_\infty(\Omega, X)^* = \mathcal{C}(\Omega, X)^*$, Theorem 4.8 immediately yields the classical Dinculeanu–Singer theorem. Notice that, for every $y^* \in Y^*$, we can identify $I_{y^*} \in \mathcal{P}_1(\mathcal{C}(\Omega), X^*) = \mathcal{C}(\Omega, X)^*$ with its (unique) representing measure $m_{y^*} : \Sigma \rightarrow X^*$. Let us stress that below we do not need to know about the Riesz–Singer representation of $\mathcal{C}(\Omega, X)^*$ as *rcabv*(Σ, X^*). However, we get the regularity of the measures m_{y^*} from our general setting. We also obtain the countable additivity of m_{y^*} thanks to the Bartle–Dunford–Schwartz theorem (because I_{y^*} are weakly compact). Moreover, the measures m_{y^*} are of bounded variation (because they are the representing measures of absolutely summing operators I_{y^*} (see, e.g., [8, p. 162, Theorem 3])). So that, in the special case when $Y = \mathbb{K}$, also the Riesz–Singer theorem is contained in Corollary 4.10 below (recall that, for a vector measure $m : \Sigma \rightarrow X^*$, one has $\|m\|_1(\Omega) = |m|(\Omega)$, the variation of m on Ω (see, e.g., [11, p. 54, Proposition 4])).

Corollary 4.10 (cf. the Dinculeanu–Singer theorem, e.g., [8, p. 182]). *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space.*

(a) *Every operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ has a unique representing measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$. This measure coincides with the representing measure of its associated operator $U^\# \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$.*

(b) *Assume that $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then, there exists an operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ such that m is its representing measure if and only if for all $y^* \in Y^*$,*

$$m_{y^*} \in \mathcal{C}(\Omega, X)^*,$$

and the map $Y^ \rightarrow \mathcal{C}(\Omega, X)^*$, $y^* \mapsto m_{y^*}$, is linear, bounded, and weak*-to-weak* continuous.*

In this case, $m_{y^} : \Sigma \rightarrow X^*$ is countably additive and of bounded variation, $m_{y^*} = U^*y^*$ for all $y^* \in Y^*$, $\|U\| = \|m\|_1(\Omega)$, and m is weakly regular.*

Remark 4.11. As we mentioned in the Introduction, in our general treatise, we did not follow any of the traditional proofs of the Dinculeanu–Singer theorem. The traditional proofs are of two types, although both extend methods of the classical proof of the Bartle–Dunford–Schwartz theorem in [2, Theorem 3.1] or [12, p. 492, Theorem 2]. The proofs, e.g., by Batt and König [4], Dinculeanu [9], [11, pp. 398–399, Theorem 9], Foiaş and Singer [14], Swong [24], Tucker [25], essentially rely on the Riesz–Singer representation theorem. The proofs, e.g., by Brooks and Lewis [5], and Diestel and Uhl [8, pp. 181–182] use “the device of embedding isometrically the simple functions in $\mathcal{C}(\Omega, X)^{**}$ and thus reducing the problem to utilizing the representing theorem for operators $L \in \mathcal{L}(\mathcal{B}(\Sigma, X), Y)$, which can be easily established”. We quoted Brooks and Lewis [5, p. 139] here; the mentioned representing theorem can be found in Dinculeanu’s book [11, p. 145, Theorem 1].

Remark 4.12. Batt and Berg [3] introduced the notion of the *weak extension* of an operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$, which is precisely the integration operator $\hat{U} \in \mathcal{L}(\mathcal{B}(\Sigma, X), Y^{**})$ with respect to the representing measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ of U . They proved that $\|\hat{U}\| = \|U\|$, $\hat{U}(\chi_E x) = m(E)x$ for all $E \in \Sigma$ and $x \in X$ (see [3, Theorem 1]), and that $\text{ran } m \subset \mathcal{L}(X, Y)$ if and only if $\text{ran } \hat{U} \subset Y$ (see [3, Theorem 2]). However, as our Theorems 3.3 and 3.4 clearly show, these are general properties of any integration operator $\hat{U} \in \mathcal{L}(\mathcal{B}(\Sigma, X), Y^{**})$ and its restriction $U := \hat{U}|_{\mathcal{C}(\Omega, X)}$. Moreover, even $\|\hat{U}\| = \|U\| = \|m\|_1(\Omega)$ and (by (9) and the “moreover” part of Theorem 3.4) $\hat{U}(\chi_E x) = m(E)x = U^{**}(\chi_E \otimes x)$ for all $E \in \Sigma$ and $x \in X$ in this general case.

5. COMPLEMENTS TO THE DINCULEANU–SINGER THEOREM

Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. Let $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. In [20], we studied the problem when does there exist an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $S = U^\#$? In this section, we shall apply some result from [20] to prove some qualitative complements to Theorem 4.8, the extension of the Dinculeanu–Singer theorem.

The idea behind the results below is as follows: the existence of an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that a given vector measure $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ is its representing measure is equivalent to the existence of an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ such that m is the representing measure of S and such that $S = U^\#$. Notice that we shall not need Theorem 4.8 at

all. Besides [20], we shall rely on Theorem 2.4, our extension of the Bartle–Dunford–Schwartz theorem, together with Theorem 4.3 and Proposition 4.4.

The next theorem also contributes to the classical Dinculeanu–Singer case when $p = \infty$.

Theorem 5.1. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. Assume that $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then, there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that m is its representing measure if and only if*

(i) *for all $x \in X$,*

$$\langle y^*, m_x(\cdot) \rangle \in \mathcal{C}(\Omega)^*, \quad y^* \in Y^*,$$

and the map $Y^ \rightarrow \mathcal{C}(\Omega)^*$, $y^* \mapsto \langle y^*, m_x(\cdot) \rangle$, is linear, bounded and weak*-to-weak* continuous, and*

(ii) *one of the following equivalent conditions holds:*

(a) *there exists a constant $c > 0$ such that, for all finite systems $(x_i)_{i=1}^n \subset X$ and $(\varphi_i)_{i=1}^n \subset \mathcal{C}(\Omega)$,*

$$\left\| \left(\int_{\Omega} \varphi_i dm_{x_i} \right) \right\|_{p'}^w \leq c \|(x_i)\|_{\infty} \|(\varphi_i)\|_{p'}^w;$$

(b) *there exists a constant $c > 0$ such that, for all $(x_i) \in \ell_{\infty}(X)$ and $(\varphi_i) \in \ell_{p'}^w(\mathcal{C}(\Omega))$, and for all $n \in \mathbb{N}$,*

$$\left\| \left(\int_{\Omega} \varphi_i dm_{x_i} \right)_{i=n}^{\infty} \right\|_{p'}^w \leq c \|(x_i)_{i=n}^{\infty}\|_{\infty} \|(\varphi_i)_{i=n}^{\infty}\|_{p'}^w;$$

(c) *if $(x_i) \in \ell_{\infty}(X)$ and $(\varphi_i) \in \ell_{p'}^w(\mathcal{C}(\Omega))$, then $(\int_{\Omega} \varphi_i dm_{x_i}) \in \ell_{p'}^w(Y)$;*

(d) *if $(x_i) \in c_0(X)$ and $(\varphi_i) \in \ell_{p'}^w(\mathcal{C}(\Omega))$ (or $(x_i) \in \ell_{\infty}(X)$ and $(\varphi_i) \in \ell_{p'}^u(\mathcal{C}(\Omega))$), then $(\int_{\Omega} \varphi_i dm_{x_i}) \in \ell_{p'}^u(Y)$.*

Proof. We are going to use the following fact. Assume that m is the representing measure of an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. Since

$$(S\varphi)x = \left(\int_{\Omega} \varphi dm \right)x = \int_{\Omega} \varphi dm_x \quad \text{for all } \varphi \in \mathcal{C}(\Omega) \text{ and } x \in X,$$

by [20, Corollary 3.4], every condition included in (ii) is equivalent to the existence of an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^{\#} = S$.

For the “only if” part, let $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ be such that m is its representing measure. By Proposition 4.4, m is also the representing measure of its associated operator $U^{\#} \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, and, by the above fact, (ii) holds; condition (i) is immediate from Theorem 2.4.

For the “if” part, condition (i) implies that there exists an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ such that m is its representing measure (see Theorem

2.4). And, by the above fact, condition (ii) implies that there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^\# = S$. Then, by Theorem 4.3, m is also the representing measure of U . \square

In the next theorem, we use [20, Corollary 2.5], which asserts that, for every operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, there exists an operator $U \in \mathcal{L}(\mathcal{C}_1(\Omega, X), Y)$ such that $U^\# = S$.

Theorem 5.2. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Assume that $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then, there exists an operator $U \in \mathcal{L}(\mathcal{C}_1(\Omega, X), Y)$ such that m is its representing measure if and only if there exists an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ such that m is its representing measure.*

Proof. The necessary condition is clear by taking $S = U^\#$ and applying Proposition 4.4. By [20, Corollary 2.5], for a given operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, there exists an operator $U \in \mathcal{L}(\mathcal{C}_1(\Omega, X), Y)$ such that $S = U^\#$. From Theorem 4.3, the sufficient condition is clear. \square

The following results, which are similar to Theorem 5.2, can be obtained using [20, Corollaries 2.6 and 2.7] (instead of [20, Corollary 2.5]).

Theorem 5.3. *Let X and Y be Banach spaces such that X^* is of cotype 2. Let Ω be a compact Hausdorff space. Assume that $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then, for every $p \leq 2$, there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that m is its representing measure if and only if there exists an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ such that m is its representing measure.*

Theorem 5.4. *Let X and Y be Banach spaces such that X^* is of cotype q , where $2 \leq q < \infty$. Let Ω be a compact Hausdorff space. Assume that $m : \Sigma \rightarrow \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then, for every $p \leq q'$, there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that m is its representing measure if and only if there exists an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ such that m is its representing measure.*

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DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE CIENCIAS EXPERIMENTALES,
UNIVERSIDAD DE HUELVA, CAMPUS UNIVERSITARIO DE EL CARMEN, 21071 HUELVA,
SPAIN

E-mail address: `fmjimenez@dmат.uhu.es`

INSTITUTE OF MATHEMATICS AND STATISTICS, UNIVERSITY OF TARTU, J. LIIVI
2, 50409 TARTU, ESTONIA; ESTONIAN ACADEMY OF SCIENCES, KOHTU 6, 10130
TALLINN, ESTONIA

E-mail address: `eve.oja@ut.ee`

DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE CIENCIAS EXPERIMENTALES,
UNIVERSIDAD DE HUELVA, CAMPUS UNIVERSITARIO DE EL CARMEN, 21071 HUELVA,
SPAIN

E-mail address: `candido@uhu.es`